Discrete Stochastic Cascade Dynamics on Finite Graphs with Irreversible Energy Depletion

Part II: Necessity of Species Asymmetry for Structural Hierarchy

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github.com/Shiv2071/Discrete-cascade-model

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Abstract

The companion paper established a discrete stochastic cascade model with guaranteed termination, emergent structure, and a four-level hierarchy. A foundational question remained open: *is the asymmetry between excitation species a necessary condition for this hierarchy, or an incidental modelling choice?*

We prove that asymmetry is necessary, within the class of models defined in Part I, in a precise, theorem-level sense. *Symmetric Collapse:* when both species share identical self-interaction coefficients and frequencies, the two-species system degenerates into a single-species annihilation process, the Landau free-energy landscape flattens, and the ordered bonding phase ceases to exist. *Differential Depletion:* the forbidden Y–Y self-interaction channel is independently necessary for bond formation; it creates an asymmetric consumption rate that concentrates the scarcer species at bond sites, providing the mechanism that the Landau functional requires. *Beat Frequency:* the frequency mismatch $\omega_X \neq \omega_Y$ is independently necessary for a sustained cascade mechanism; the phase-interference factor makes the cross-channel efficiency oscillate with beat period $2\pi/|\omega_X - \omega_Y|$, giving the ripple a recurrent deterministic drive with explicit amplitude $A_{\text{beat}} = (1-\vartheta)\sin(|\omega_X - \omega_Y|/2)\,\gamma_1 \alpha_{XY} \omega_X \omega_Y \bar{X}\bar{Y}$; at equal frequencies the drive vanishes identically and only decaying transients remain.

We further introduce an *asymmetry index* $\mathcal{A}$ that reduces the combined deviation from species symmetry to a single scalar, derive an explicit critical threshold $\mathcal{A}_{\text{crit}}$ (in closed form for the frequency factor) below which the ripple cannot reach the explosive regime, and establish a monotone bound on hierarchy depth as a function of $\mathcal{A}$. The results show that the structural richness of the discrete cascade model is not robust to symmetrisation: within this model class, asymmetry is a necessary structural condition, not merely a parameter choice.

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1 Introduction

1.1 Context

In the companion paper Goswami [2026] (hereafter Paper I), we introduced a discrete stochastic cascade model on finite graphs with two asymmetric excitation species (X and Y), a non-renewable capacity energy, threshold-driven regime switching, and Landau free-energy bond formation. The model was shown to possess three rigorous properties: energy monotonicity, finite total activity, and almost sure absorption.

Paper I identified seven open problems. The seventh asked:

> *What happens in the symmetric limit $\alpha_{XX} = \alpha_{YY}$, $\omega_X = \omega_Y$? Does the structural hierarchy collapse? The forbidden Y–Y channel and the frequency mismatch are constitutive assumptions; understanding their necessity (or relaxation) is an important structural question.*

This paper resolves that question completely. The answer is: **yes, the hierarchy collapses, and both asymmetries are independently necessary for this model class.**

1.2 Summary of Results

We prove three main theorems:

1. **Symmetric Collapse** (theorem 3.2): When both species have identical self-interaction coefficients and identical intrinsic frequencies, the two-species model reduces to an effective single-species annihilation process. The Landau free energy for bond formation degenerates, and the ordered phase (bonding) ceases to exist.

2. **Differential Depletion** (theorem 4.2): The forbidden Y–Y channel ($\alpha_{YY} = 0$) creates an asymmetric depletion rate: X is consumed faster than Y (via both X–X and X–Y channels, versus Y's single X–Y channel). This differential is the driving mechanism for bond formation: the scarcer species (X) becomes the bottleneck, concentrating co-located X–Y pairs at sites where bonds can form.

3. **Beat Frequency** (theorem 5.3): Through the phase-interference factor, the frequency mismatch $\omega_X \neq \omega_Y$ generates a literal beat of period $2\pi/|\omega_X - \omega_Y|$ in the cross-channel efficiency, hence in the structural increment. This beat is the sole *recurrent, deterministic* source of oscillation in the structural state S, which the ripple measure F detects. Without it, the deterministic drive vanishes identically and only decaying transients and Poisson noise remain: the system cannot sustain cascades.

We additionally derive:

- An *asymmetry index* $\mathcal{A}$ that captures the combined deviation from species symmetry in a single scalar (theorem 6.1).
- A *critical asymmetry threshold* $\mathcal{A}_{\text{crit}}$, with a closed-form expression for its frequency factor, below which the ripple cannot reach the explosive regime (theorem 6.3).
- A monotone bound on hierarchy depth as a function of $\mathcal{A}$ (theorem 6.6).

1.3 Organisation

Section 2 recalls the model from Paper I and establishes notation. Section 3 analyses the symmetric limit. Section 4 proves necessity of the forbidden Y–Y channel. Section 5 proves necessity of the frequency mismatch. Section 6 defines the asymmetry index and derives the hierarchy-depth bound. Section 8 discusses implications.

Scope of claims. All necessity results in this paper are proved for the specific model class defined in Paper I. We show that within this class, both asymmetries are required for the structural hierarchy; we do not claim that no alternative model could achieve similar structure by other mechanisms.

2 Setup and Notation

We adopt all definitions, parameters, and notation from Paper I. For convenience we recall the key elements.

2.1 Model Summary

The system lives on a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, $|\mathcal{V}| = P$. Each vertex carries state $(X, Y, \mathcal{E}, S)$. The update rule proceeds as: interaction $\to$ ripple $\to$ regime $\to$ bonds $\to$ energy $\to$ structure $\to$ diffusion.

Ripple intensity is the discrete second difference of the structural state: $F(p,n) = |S(p,n) - 2\,S(p,n{-}1) + S(p,n{-}2)| = |\Delta^2 S(p,n)|$. Total capacity energy obeys $\mathcal{E}_{\text{tot}}(n+1) = \mathcal{E}_{\text{tot}}(n) - \mathcal{D}(n)$ with $\mathcal{D}(n) \geq 0$ (Paper I: Energy Monotonicity); strict decrease whenever the system is active.

2.2 Interaction Rates

Each species carries an intrinsic phase $\phi_X(n) = \omega_X n$, $\phi_Y(n) = \omega_Y n$, and cross-interaction efficiency is modulated by phase alignment through the *interference factor* (Paper I):

$$\Theta(n) = \vartheta + (1-\vartheta)\,\frac{1+\cos\big((\omega_X - \omega_Y)\,n\big)}{2} \;\in\; [\vartheta, 1]\,, \qquad \vartheta \in (0,1]\,, \tag{1}$$

with coherence floor ϑ. Same-species pairs are always phase-aligned, so the self-channels are unmodulated:

$$R_{XY}(p,n) = \alpha_{XY}\,\omega_X\,\omega_Y\,\Theta(n)\,X(p,n)\,Y(p,n)\,, \tag{2}$$

$$R_{XX}(p,n) = \alpha_{XX}\,\omega_X^2\,\frac{X(p,n)(X(p,n)-1)}{2}\,, \tag{3}$$

$$R_{YY}(p,n) = \alpha_{YY}\,\omega_Y^2\,\frac{Y(p,n)(Y(p,n)-1)}{2}\,. \tag{4}$$

When $\omega_X = \omega_Y$, the interference factor is identically 1 and all formulas reduce to their unmodulated forms.

In Paper I, $\alpha_{YY} = 0$ (forbidden Y–Y channel). In this paper we consider the general case $\alpha_{YY} \geq 0$ to study what happens when the constraint is relaxed.

2.3 Depletion Rates

Definition 2.1 (Per-species expected depletion rate)**.** The expected number of X and Y excitations consumed per step at vertex p is:

$$\mu_X(p,n) = \mathbb{E}[N_{XY}(p,n)] + 2\,\mathbb{E}[N_{XX}(p,n)] + \mathbb{E}[B(p,n)]\,, \tag{5}$$
$$\mu_Y(p,n) = \mathbb{E}[N_{XY}(p,n)] + 2\,\mathbb{E}[N_{YY}(p,n)] + \mathbb{E}[B(p,n)]\,. \tag{6}$$

The factor of 2 arises because each self-interaction annihilates two excitations of the same species.

Definition 2.2 (Depletion asymmetry)**.**

$$\delta\mu(p,n) = \mu_X(p,n) - \mu_Y(p,n) = 2\,\mathbb{E}[N_{XX}(p,n)] - 2\,\mathbb{E}[N_{YY}(p,n)]\,.$$

When $\alpha_{YY} = 0$ (Paper I model), $\delta\mu = 2\,\mathbb{E}[N_{XX}] \geq 0$: X always depletes at least as fast as Y. A typical simulation run (Paper I dynamics) illustrates this: fig. 1 shows total X and Y over time; X and Y diverge as the depletion gap accumulates.

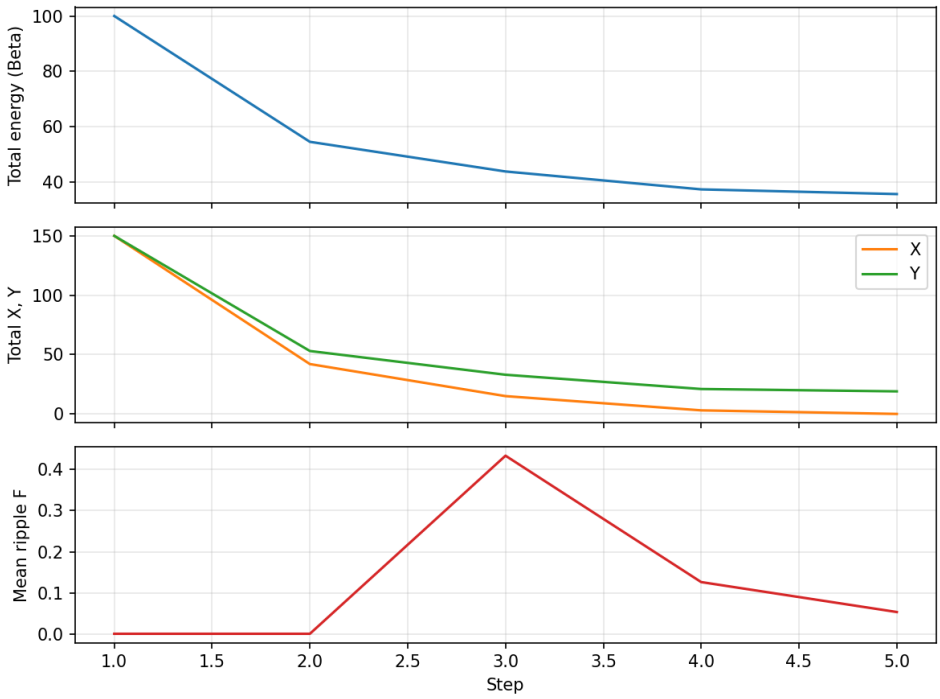

Figure 1: Simulation run (Paper I model, $P = 50$ sites): total energy (top), total X and Y (middle), mean ripple F (bottom). The middle panel illustrates differential depletion: X and Y evolve asymmetrically.

2.4 Structural Increment

$$\Delta S(p,n) = S(p,n{+}1) - S(p,n) = \gamma_1\, N_{XY}(p,n) + \gamma_{XX}\, N_{XX}(p,n) + \gamma_{YY}\, N_{YY}(p,n) + \gamma_2\, B(p,n)\,, \tag{7}$$

where we include a hypothetical γ_{YY} for the generalised model.

3 The Symmetric Limit

Definition 3.1 (Symmetric model)**.** The *symmetric model* is obtained by setting:

$$\alpha_{XX} = \alpha_{YY} = \alpha_s\,, \qquad \omega_X = \omega_Y = \omega\,, \qquad \gamma_{XX} = \gamma_{YY} = \gamma_s\,. \tag{8}$$

Theorem 3.2 (Symmetric Collapse)**.** *Under the symmetric conditions* (8) *with symmetric initial data* ($X(p,0) \overset{d}{=} Y(p,0)$ *for all* p)*, the following hold:*

(a) ***Species indistinguishability.*** *The joint process* $(X(p,n), Y(p,n))$ *is exchangeable:*

$$(X(p,n), Y(p,n)) \overset{d}{=} (Y(p,n), X(p,n)) \qquad \forall\, p,\, n\,.$$

(b) ***Single-species reduction.*** *Define the total excitation count* $Z(p,n) = X(p,n) + Y(p,n)$*. The total annihilation rate at* p *satisfies*

$$R_{\text{tot}}(p,n) = \big(\alpha_{XY} - \alpha_s\big)\omega^2\, XY + \frac{\alpha_s\,\omega^2}{2}\, Z(Z-1)\,,$$

which depends on the configuration only through Z *up to the cross-term* $(\alpha_{XY} - \alpha_s)\,\omega^2 XY$*; since* $XY \le Z^2/4$*, this rate is bounded above by* $[(\alpha_{XY} + \alpha_s)\,\omega^2/2]\, Z(Z-1)$ *for* $Z \ge 2$*. In the* fully symmetric *case* $\alpha_{XY} = \alpha_s$*, the cross-term vanishes identically and* Z *evolves autonomously as a single-species annihilation–diffusion process with exact effective rate*

$$R_Z(p,n) = \frac{\alpha_s\,\omega^2}{2} \cdot Z(p,n)\big(Z(p,n) - 1\big)\,.$$

(c) ***Degenerate order parameter.*** *The bond order parameter* $\psi = \sqrt{X \cdot Y}$ *satisfies* $\mathbb{E}[\psi^2] = \mathbb{E}[X \cdot Y] = \mathbb{E}[X^2]$ *(since* $X \overset{d}{=} Y$*). The Landau free energy* $\mathcal{F}(\psi) = a_0(T_{\text{eff}} - T_c)\psi^2 + b\,\psi^4$ *reduces to a free energy in the single variable* X*, and the order–disorder distinction between "bonded" and "free" populations vanishes: the system cannot distinguish a bond from a self-interaction.*

(d) ***No structural hierarchy.*** *The hierarchy levels (free excitation* $\to$ *co-located pair* $\to$ *bond* $\to$ *clump) collapse:*

- *Level 0 and Level 1 merge (every pair of same-type excitations is also a valid interacting pair).*
- *Level 2 (bonds) cannot be distinguished from self-interactions (both consume two excitations of identical species).*
- *Level 3 (clumps) reduces to generic density fluctuations with no internal structure.*

Proof. **(a)** Under the symmetric conditions, the transition kernel is invariant under the permutation $X \leftrightarrow Y$. Since the initial distribution is also invariant, exchangeability follows by induction on n.

(b) By exchangeability, $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Z]/2$. The total annihilation rate from all three channels is:

$$\underbrace{\alpha_{XY}\,\omega^2\, X\,Y}_{\text{cross}} + \underbrace{\alpha_s\,\omega^2\,\frac{X(X-1)}{2}}_{\text{X–X}} + \underbrace{\alpha_s\,\omega^2\,\frac{Y(Y-1)}{2}}_{\text{Y–Y}}$$
$$= \alpha_{XY}\,\omega^2\, X\,Y + \alpha_s\,\omega^2\,\frac{X^2 + Y^2 - X - Y}{2}\,.$$

Using $X^2 + Y^2 = (X+Y)^2 - 2XY = Z^2 - 2XY$ and $X + Y = Z$:

$$= \big(\alpha_{XY} - \alpha_s\big)\omega^2\, XY + \frac{\alpha_s\,\omega^2}{2}\big(Z^2 - Z\big)\,.$$

This establishes the stated identity for R_{tot}. For the upper bound: by exchangeability, $\mathbb{E}[XY \mid Z] = \mathbb{E}[X(Z-X) \mid Z] = Z\mathbb{E}[X \mid Z] - \mathbb{E}[X^2 \mid Z]$. Since $X \mid Z$ is symmetric about $Z/2$, we have $\mathbb{E}[X \mid Z] = Z/2$ and hence $\mathbb{E}[XY \mid Z] = Z^2/4 - \mathrm{Var}(X \mid Z) \le Z^2/4$, with equality only when $X = Y = Z/2$ deterministically. Substituting yields the stated bound $[(\alpha_{XY} + \alpha_s)\omega^2/2]\, Z(Z-1)$ for $Z \ge 2$ (when $\alpha_{XY} < \alpha_s$ the cross-term is non-positive and the bound is immediate).

For the exact reduction: when $\alpha_{XY} = \alpha_s$, the cross-term vanishes identically (not merely in expectation), so $R_{\text{tot}} = (\alpha_s\omega^2/2)\, Z(Z-1)$ is a function of Z alone. Since the diffusion step acts identically on both species ($D_X = D_Y$), the pair (X, Y) influences the law of Z only through Z itself, and Z is Markov with the stated rate. When $\alpha_{XY} \neq \alpha_s$, the rate retains a residual dependence on XY, so Z alone is not autonomous; the reduction then holds as a rate bound rather than an equality in law.

(c) By exchangeability, $\mathbb{E}[\psi^2] = \mathbb{E}[XY]$. Since X and Y are identically distributed:

$$\mathbb{E}[XY] = \mathbb{E}[X]^2 + \mathrm{Cov}(X, Y)\,.$$

The order parameter ψ no longer distinguishes "cross-species co-presence" from "same-species density": both scale identically with the single-field variable Z. The Landau free energy $\mathcal{F}(\psi)$ becomes $\mathcal{F}(Z/2)$, a function of the total density alone. There is no phase transition between an "unbonded" and "bonded" state because the two are indistinguishable.

(d) Without a well-defined bonding transition, the hierarchy collapses:

- A "bond" between two identical excitations is operationally indistinguishable from the pairwise annihilation that already defines the self-interaction channel.
- The structural state S accumulates contributions from all channels equally (since $\gamma_1 = \gamma_s = \gamma_{YY}$), producing a spatially smooth increment field with no preferential concentration.
- Without concentration, there are no clumps; only uniform density decay. □

Remark 3.3. The symmetric collapse is not a gradual degradation. It is a qualitative change: the four-level hierarchy of Paper I (excitation → pair → bond → clump) reduces to a single level (excitation → annihilation). The entire structural emergence layer is absent.

Corollary 3.4 (Reduction to single-species annihilation)**.** *Under the* fully symmetric *conditions* (8) *together with $\alpha_{XY} = \alpha_s$ (indistinguishable species cannot carry distinct cross- and self-interaction rates), with symmetric initial data, the dynamics of the total excitation count $Z(p,n) = X(p,n) + Y(p,n)$ are equivalent in law to a single-species diffusion-limited annihilation process $Z + Z \to \varnothing$ on the same graph, with effective annihilation rate $R_Z = (\alpha_s\,\omega^2/2)\, Z(Z-1)$ and diffusion coefficient $D_X = D_Y$. For general $\alpha_{XY} \neq \alpha_s$ the reduction holds as the rate bound of theorem 3.2(b), and the hierarchy collapse in parts (c)–(d) below follows from exchangeability alone. In particular:*

(i) *The energy monotonicity, finite activity, and almost sure absorption guarantees of Paper I still hold for the reduced process.*

(ii) *The Landau bond-formation mechanism is inoperative (no order–disorder transition exists).*

(iii) *The model reduces exactly to the classical $A + A \to \varnothing$ annihilation–diffusion system [Toussaint & Wilczek, 1983] on finite graphs with a finite energy budget.*

Proof. Part (i) follows because the proofs of energy monotonicity, finite activity, and absorption in Paper I depend only on the non-negativity of depletion terms, which hold regardless of symmetry. Part (ii) follows from theorem 3.2(c): the order parameter ψ becomes a function of the total density Z alone, and the Landau free energy loses its order–disorder distinction. Part (iii) follows from theorem 3.2(b): under $\alpha_{XY} = \alpha_s$ the cross-term vanishes identically, Z is Markov, and the effective rate $R_Z \propto Z(Z-1)$ is the standard pairwise annihilation rate on the total species count. □

4 Necessity of the Forbidden Y–Y Channel

We now show that the specific asymmetry $\alpha_{YY} = 0$ (rather than just any asymmetry) plays a distinguished role.

4.1 Differential Depletion Mechanism

Lemma 4.1 (Depletion gap)**.** *In the Paper I model ($\alpha_{YY} = 0$, $\alpha_{XX} > 0$), for any vertex p with $X(p,n) \geq 2$ and $Y(p,n) \geq 1$:*

$$\delta\mu(p,n) = 2\,\mathbb{E}[N_{XX}(p,n)] = \alpha_{XX}\,\omega_X^2\,X(p,n)\big(X(p,n)-1\big) > 0\,.$$

X depletes strictly faster than Y.

Proof. From theorem 2.2 with $\alpha_{YY} = 0$: $\delta\mu = 2\mathbb{E}[N_{XX}] - 0 = 2\mathbb{E}[N_{XX}]$. Since $X \geq 2$ and $\alpha_{XX} > 0$, the Poisson rate $R_{XX} > 0$, hence $\mathbb{E}[N_{XX}] > 0$. □

Theorem 4.2 (Differential Depletion)**.** *In the Paper I model, the depletion gap produces the following effects:*

(a) ***Species ratio divergence.*** *At the interaction step, the ratio of expected post-interaction counts at any active vertex exceeds $Y(p,n)/X(p,n)$. Since diffusion conserves the graph totals $\sum_p X$ and $\sum_p Y$, the global ratio of expected totals is non-decreasing and strictly increases whenever some vertex is active. Y becomes the majority species.*

(b) ***Minority bottleneck.*** *Bond formation is limited by the minority species in two ways. First, the bond count is capped directly:*

$$B(p,n) \leq \min\big(X(p,n),\, Y(p,n)\big)\,,$$

since each bond consumes one X and one Y (bond-count definition, Paper I). Second, the bond order parameter $\psi = \sqrt{X \cdot Y} = \sqrt{X} \cdot \sqrt{Y}$ vanishes as $X \to 0$ regardless of the abundance of Y, and grows only as $\sqrt{X}$ at fixed Y. Since X is the scarcer species, bond formation becomes X-limited*: bonds form only where X persists.*

(c) ***Spatial concentration.*** *Because X is scarcer and consumed faster, X excitations that survive are concentrated at vertices where self-interaction is locally low (due to stochastic fluctuation). This concentrates the order parameter ψ at sparse, specific sites (the precursor to bond sites).*

(d) ***Bond formation as condensation.*** *Bond formation at a vertex requires co-located X and Y with $T_{\text{eff}} < T_c$ (low ripple, ordered phase). Since X is rare, co-location events are spatially sparse. When they occur, the high Y density ensures the cross-interaction rate is large, driving S and the ripple dynamics. This is a* condensation *of the minority species at bond sites.*

Proof. **(a)** Let $r(p,n) = Y(p,n)/X(p,n)$ for $X > 0$. In expectation:

$$\mathbb{E}[X(p,n{+}1)] \approx X - \mathbb{E}[N_{XY}] - 2\mathbb{E}[N_{XX}] = X - \mu_X\,,$$
$$\mathbb{E}[Y(p,n{+}1)] \approx Y - \mathbb{E}[N_{XY}] = Y - \mu_Y\,.$$

Since $\mu_X > \mu_Y$ (theorem 4.1), the ratio of *expected* values increases:

$$\frac{\mathbb{E}[Y(n{+}1)]}{\mathbb{E}[X(n{+}1)]} = \frac{Y-\mu_Y}{X-\mu_X} > \frac{Y}{X} = r(p,n)\,,$$

provided $X - \mu_X > 0$ (activity persists). Note that this is a statement about the ratio of expectations, not the expectation of the ratio; the latter would require $f(x) = 1/x$ to be convex (which it is for $x > 0$), giving $\mathbb{E}[Y/X] \geq \mathbb{E}[Y]/\mathbb{E}[X]$ by Jensen's inequality, so the expectation of the ratio diverges at least as fast. The computation above concerns the interaction step at a single vertex. Diffusion exchanges excitations between neighbouring vertices but creates and destroys none, so summing over all vertices, the graph totals obey $\mathbb{E}[\sum_p X(p,n{+}1)] = \sum_p (X - \mu_X)$ and $\mathbb{E}[\sum_p Y(p,n{+}1)] = \sum_p (Y - \mu_Y)$; the same inequality then holds for the ratio of expected totals, which is the global statement in (a), free of any diffusion approximation.

(b) The cap $B(p,n) \leq \min(X,Y)$ is immediate from the bond-count definition of Paper I: the number of bonds formed at a vertex is explicitly truncated at $\min(X,Y)$ because each bond consumes one excitation of each species. For the order parameter, write $\psi = \sqrt{XY} = \sqrt{X}\,\sqrt{Y}$. For any bounded local Y count ($Y \leq Y_{\max}$), $\psi \leq \sqrt{Y_{\max}}\,\sqrt{X}$, so $\psi = O(\sqrt{X})$ and $\psi = 0$ whenever $X = 0$. We note that ψ itself is *not* bounded by $\min(X,Y)$ (the geometric mean lies between the minimum and the maximum); the bottleneck operates through the bond-count cap and the $\sqrt{X}$ scaling, both of which vanish with the minority species.

(c) The X depletion has two channels: cross-interaction (rate $\propto XY$, present everywhere with Y) and self-interaction (rate $\propto X^2$, strong where X is locally dense). Sites with locally low X density have lower self-interaction rates, so X survives preferentially at low-density sites. This creates spatial inhomogeneity in X even from uniform initial conditions.

(d) At sites where X survives (sparse), Y is abundant (by part (a)). The co-density XY is maximised at these sites. The Landau bond-formation rule of Paper I operates in the ordered phase ($T_{\text{eff}} < T_c$) and requires a non-zero order parameter $\psi = \sqrt{XY} > 0$, with the realised bond count capped by $\min(X,Y)$; both the trigger and the cap are most favourable where the co-density is highest, so bonds form preferentially at these concentrated sites. □

Corollary 4.3 (Necessity of forbidden Y–Y). *If $\alpha_{YY} > 0$, then $\delta\mu = 2(\mathbb{E}[N_{XX}] - \mathbb{E}[N_{YY}])$, which can be zero or negative depending on local densities. In particular, if $\alpha_{YY} = \alpha_{XX}$ (symmetric self-interaction), $\delta\mu = 0$ when $X = Y$, and the differential depletion mechanism is destroyed. The species ratio does not diverge, no minority bottleneck forms, and the spatial concentration that drives bond formation vanishes.*

Remark 4.4. The forbidden Y–Y channel is not merely a simplification. It is the *origin* of the species asymmetry that creates the structural hierarchy. Allowing Y–Y interactions, even weakly, degrades the depletion gap and reduces hierarchy depth (see section 6).

5 Necessity of Frequency Mismatch

Remark 5.1 (Envelope decomposition)**.** Throughout this section we analyse the *deterministic envelope* of the structural increment: the conditional expectation $\mathbb{E}[\Delta S(p,n) \mid \mathcal{F}_n]$, viewed as a function of the step index through the interference factor $\Theta(n)$ and through the slowly varying excitation counts. Realisations fluctuate around this envelope with Poisson noise whose scale decays with the interaction rates; the envelope isolates the *systematic* drive on the ripple, which is the object of the necessity claim. Statements about the envelope are exact; the passage from envelope to realised ripple adds a decaying noise floor that we quantify where needed.

5.1 Interaction Rates and Structural Beating

By (2)–(4), the structural increment at vertex p has conditional expectation

$$\mathbb{E}[\Delta S(p,n) \mid \mathcal{F}_n] = \Theta(n)\, g(p,n) + h(p,n)\,, \tag{9}$$

where

$$g(p,n) = \gamma_1\, \alpha_{XY}\, \omega_X\, \omega_Y\, X\, Y\,, \qquad h(p,n) = \gamma_{XX}\, \alpha_{XX}\, \omega_X^2\, \frac{X(X-1)}{2} + \gamma_2\, \mathbb{E}[B(p,n) \mid \mathcal{F}_n]\,.$$

The cross-channel envelope carries the multiplicative oscillation $\Theta(n)$; the self-channel and bond terms are unmodulated. Write $\Delta\omega = \omega_X - \omega_Y$ for the frequency mismatch.

Lemma 5.2 (Beat increment)**.** *The one-step increment of the interference factor is*

$$\Theta(n) - \Theta(n{-}1) = -(1-\vartheta)\, \sin\Big(\Delta\omega\, \big(n - \tfrac{1}{2}\big)\Big)\, \sin\Big(\frac{\Delta\omega}{2}\Big)\,,$$

so that

$$\max_n \big|\Theta(n) - \Theta(n{-}1)\big| = (1-\vartheta)\, \sin\Big(\frac{|\Delta\omega|}{2}\Big) \qquad (0 \le |\Delta\omega| \le 2\pi)\,,$$

attained twice per beat period $2\pi/|\Delta\omega|$. In particular the increment vanishes identically if and only if $\Delta\omega = 0$.

Proof. Apply the product-to-sum identity $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$ with $A = \Delta\omega\, n$, $B = \Delta\omega\,(n-1)$ to the definition (1). The factor $\sin(\Delta\omega(n - \frac{1}{2}))$ sweeps through values arbitrarily close to ± 1 as n ranges over a beat period, giving the stated maximum. □

5.2 Ripple Generation via Beat Frequency

Recall that the ripple can be written in terms of first differences of the increment: $F(p,n) = |\Delta S(p,n{-}1) - \Delta S(p,n{-}2)|$.

Theorem 5.3 (Beat Frequency Theorem)**.** *Let p be a vertex with co-located excitations and let $\bar{X}, \bar{Y}$ denote characteristic excitation counts over a beat period (formally: bounds $X(p,n)Y(p,n) \geq \bar{X}\bar{Y}$ holding on the steps considered). Then:*

(a) ***Frequency matching removes the deterministic drive.*** *When $\omega_X = \omega_Y$, the interference factor is identically 1 and the envelope of $\Delta S(p,n)$ equals $g(p,n) + h(p,n)$, which changes between consecutive steps only through depletion of the excitation counts. If the per-step relative depletion is bounded by ε (slow-depletion regime), the deterministic part of the ripple satisfies*

$$F_{\text{det}}(p,n) \;\leq\; \varepsilon \left(g(p,n) + h(p,n)\right),$$

which decays with the envelope. There is no recurring deterministic mechanism that drives F toward the explosion threshold.

(b) ***Frequency mismatch generates a persistent beat.*** *When $\omega_X \neq \omega_Y$, the deterministic part of the ripple contains the term*

$$\left|\Theta(n{-}1) - \Theta(n{-}2)\right| \, g(p,n{-}1)\,,$$

which by theorem 5.2 oscillates with beat period $2\pi/|\Delta\omega|$ and returns to its maximum twice per period for as long as co-located excitations persist. The drive is recurrent*, not transient: phase realignment re-excites the ripple on every beat cycle.*

(c) ***Explicit beat amplitude.*** *The peak deterministic ripple contributed by the beat is*

$$A_{\text{beat}} = (1-\vartheta)\, \sin\!\Big(\frac{|\Delta\omega|}{2}\Big)\, \gamma_1\, \alpha_{XY}\, \omega_X\, \omega_Y\, \bar{X}\, \bar{Y}\,, \tag{10}$$

up to a depletion-drift correction of relative size $O(\varepsilon)$. $A_{\text{beat}} = 0$ if and only if $\omega_X = \omega_Y$ (within one beat aliasing period, $|\Delta\omega| < 2\pi$).

Proof. Using $F(p,n) = |\Delta S(p,n{-}1) - \Delta S(p,n{-}2)|$ and the envelope decomposition (9), the deterministic part of the ripple is

$$\begin{aligned} F_{\text{det}}(p,n) &= \left|\Theta(n{-}1)\, g(n{-}1) - \Theta(n{-}2)\, g(n{-}2) + h(n{-}1) - h(n{-}2)\right| \\ &= \Big|\underbrace{\left[\Theta(n{-}1) - \Theta(n{-}2)\right] g(n{-}1)}_{\text{beat drive}} + \underbrace{\Theta(n{-}2)\left[g(n{-}1) - g(n{-}2)\right] + \left[h(n{-}1) - h(n{-}2)\right]}_{\text{depletion drift}}\Big|\,, \end{aligned}$$

suppressing the vertex argument.

(a) If $\Delta\omega = 0$ then $\Theta \equiv 1$ by (1) and the beat drive vanishes identically. The remaining terms are differences of g and h between consecutive steps. Both g and h are polynomial in the excitation counts; if each count changes by at most a relative ε per step, then $|g(n{-}1) - g(n{-}2)| \leq 2\varepsilon\, g$ and similarly for h (to first order in ε), giving the stated bound. Because g and h decay with depletion (energy monotonicity and the finite activity bound of Paper I), this deterministic residue decays to zero and cannot recur.

(b) If $\Delta\omega \neq 0$, theorem 5.2 gives the beat drive amplitude $(1-\vartheta)\sin(|\Delta\omega|/2)\, g(n-1)$, attained twice per beat period. The drive persists precisely as long as $g(p,n) = \gamma_1 \alpha_{XY} \omega_X \omega_Y XY > 0$, i.e. as long as co-located pairs remain. Unlike the depletion drift, the beat drive does not decay between successive maxima except through the slow decay of g itself.

(c) Substituting the characteristic co-density bound $XY \geq \bar{X}\bar{Y}$ into the beat-drive term yields (10); the depletion drift contributes the stated $O(\varepsilon)$ relative correction. By theorem 5.2, the prefactor $\sin(|\Delta\omega|/2)$ vanishes exactly at $\Delta\omega = 0$ and is strictly positive for $0 < |\Delta\omega| < 2\pi$. □

Corollary 5.4 (Frequency mismatch is necessary for recurrent cascades)**.** *When $\omega_X = \omega_Y$, the only sources of ripple are the start-up transient (the onset of structural growth from $S \equiv 0$) and Poisson fluctuations, both of which decay with the depletion envelope and neither of which recurs. When $\omega_X \neq \omega_Y$ and $A_{\text{beat}} > C + \Delta$, the deterministic beat drive alone pushes F into the explosive regime, and does so recurrently (twice per beat period) for as long as sufficient co-density persists. Frequency mismatch is therefore necessary for a* sustained, recurrent *cascade mechanism, as opposed to isolated fluctuation-triggered events.*

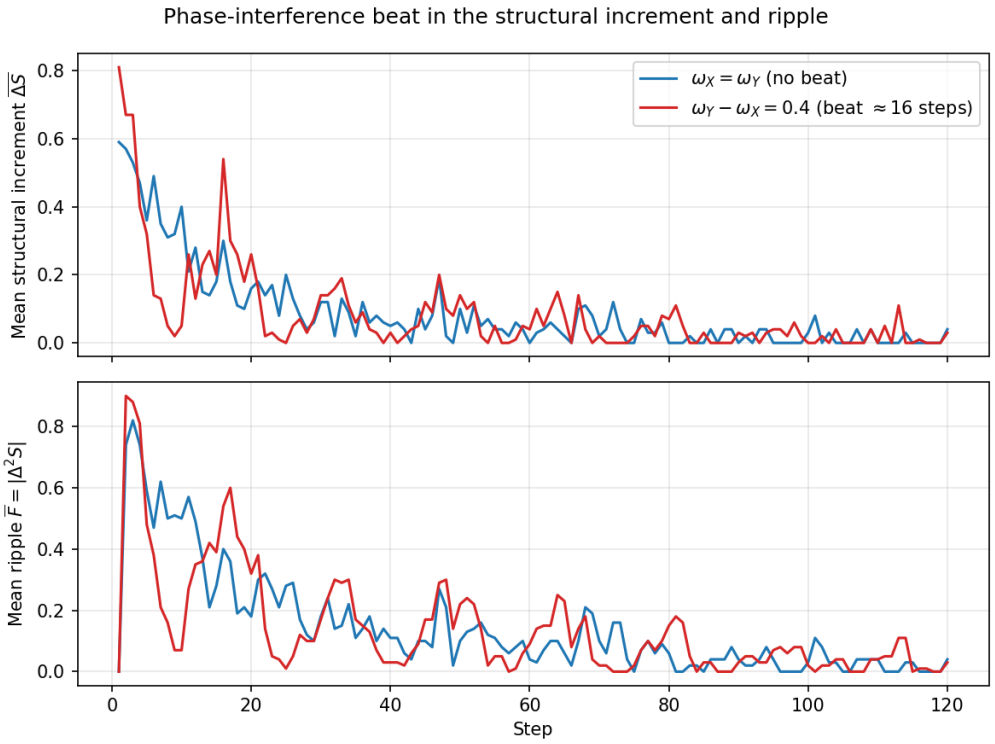

Figure 2: Beat demonstration (two runs with identical parameters and seed, differing only in ω_Y). **Top:** mean structural increment $\overline{\Delta S}$; with mismatch $\Delta\omega = 0.4$ (red) the increment carries the multiplicative beat of period $2\pi/0.4 \approx 16$ steps, absent at equal frequencies (blue). **Bottom:** mean ripple $\overline{F}$; the mismatched run re-excites the ripple on every beat cycle while the equal-frequency run decays with the depletion envelope. Generated by `model_simulation/run_beat_demo.py`.

6 Asymmetry Index and Hierarchy Depth

We now unify the two asymmetry sources into a single measure.

6.1 Asymmetry Index

Definition 6.1 (Asymmetry index)**.** The *asymmetry index* of the model is:

$$\mathcal{A} = \underbrace{\frac{|\alpha_{XX} - \alpha_{YY}|}{\alpha_{XX} + \alpha_{YY} + \epsilon}}_{\text{channel asymmetry } \mathcal{A}_c} \times \underbrace{\frac{|\omega_X - \omega_Y|}{\omega_X + \omega_Y}}_{\text{frequency asymmetry } \mathcal{A}_f}, \tag{11}$$

where $\epsilon > 0$ is a regulariser (set to α_{XY} to avoid division by zero when $\alpha_{XX} = \alpha_{YY} = 0$). The absolute value in $\mathcal{A}_c$ reflects that either species may play the minority role: if $\alpha_{YY} > \alpha_{XX}$, the differential-depletion mechanism of section 4 operates with the roles of X and Y exchanged.

Properties:

- $\mathcal{A} \geq 0$, with $\mathcal{A} = 0$ if and only if at least one asymmetry is absent ($\alpha_{XX} = \alpha_{YY}$ or $\omega_X = \omega_Y$). In particular $\mathcal{A} = 0$ for the symmetric model of theorem 3.1.
- $\mathcal{A} < 1$, with $\mathcal{A} \to 1$ in the joint limit of full channel asymmetry with dominant self-interaction ($\alpha_{YY} = 0$, $\alpha_{XX} \gg \epsilon$) and maximal frequency separation ($\omega_Y/\omega_X \to 0$ or $\omega_X/\omega_Y \to 0$).
- $\mathcal{A}$ is the product of two independent asymmetry factors, reflecting the fact that *both* asymmetries are needed (either factor being zero kills $\mathcal{A}$).

Remark 6.2. The multiplicative structure of $\mathcal{A}$ encodes the main result of this paper: channel asymmetry alone ($\mathcal{A}_c > 0$, $\mathcal{A}_f = 0$) or frequency asymmetry alone ($\mathcal{A}_c = 0$, $\mathcal{A}_f > 0$) gives $\mathcal{A} = 0$, and the hierarchy collapses in both cases. Both factors must be non-zero simultaneously.

6.2 Critical Asymmetry Threshold

Theorem 6.3 (Critical frequency asymmetry: explicit threshold)**.** *Fix the channel structure of Paper I ($\alpha_{YY} = 0$, $\alpha_{XX} > 0$, so $\mathcal{A}_c > 0$) and let $\bar{\rho} = \bar{X}\bar{Y}$ be the characteristic sustained co-density of theorem 5.3. Define*

$$\mathcal{A}_f^{\text{crit}} = \frac{2}{\omega_X + \omega_Y} \arcsin\left(\min\left(1, \frac{C + \Delta}{(1 - \vartheta)\,\gamma_1\,\alpha_{XY}\,\omega_X\,\omega_Y\,\bar{\rho}}\right)\right), \tag{12}$$

obtained by solving $A_{\text{beat}} = C + \Delta$ in (10) with $|\Delta\omega| = \mathcal{A}_f\,(\omega_X + \omega_Y)$. Then, at the level of the deterministic envelope:

(a) If $\mathcal{A}_f < \mathcal{A}_f^{\text{crit}}$, the beat drive never reaches the explosion threshold: $F_{\text{det}}(p, n) < C + \Delta$ for all p, n (up to the $O(\varepsilon)$ depletion drift). No envelope-driven explosions occur; bond formation may still proceed where ψ suffices, but without cascade-driven spatial concentration.

(b) *If $\mathcal{A}_f > \mathcal{A}_f^{\text{crit}}$ (and the arcsine in (12) is not saturated), the beat drive alone pushes F above $C + \Delta$ at every vertex where the co-density reaches $\bar{\rho}$, twice per beat period: cascades recur, spatial concentration develops, and the full four-level hierarchy can emerge.*

The combined critical index is $\mathcal{A}_{\text{crit}} = \mathcal{A}_c \cdot \mathcal{A}_f^{\text{crit}}$ evaluated at the Paper I channel structure ($\mathcal{A}_c = \alpha_{XX}/(\alpha_{XX} + \alpha_{XY})$ by theorem 6.1 with $\alpha_{YY} = 0$, $\epsilon = \alpha_{XY}$); the channel factor enters implicitly through $\bar{\rho}$, since sustained co-density requires the differential-depletion mechanism of section 4.

Proof. Both parts follow from theorem 5.3(c) by direct inversion. The beat amplitude (10) is strictly increasing in $|\Delta\omega|$ on $(0, \pi]$ through the factor $\sin(|\Delta\omega|/2)$, and $|\Delta\omega| = \mathcal{A}_f(\omega_X + \omega_Y)$ by the definition of $\mathcal{A}_f$. Setting $A_{\text{beat}} = C + \Delta$ and solving for $\mathcal{A}_f$ yields (12). Below the threshold, the peak deterministic drive falls short of $C + \Delta$; above it, the drive exceeds $C + \Delta$ at the two phase-slip steps of each beat period wherever $XY \geq \bar{\rho}$. Poisson fluctuations around the envelope widen the supercritical region (fluctuations can trigger explosions slightly below threshold) but cannot create a recurrent drive below it; the envelope statement is therefore the sharp deterministic boundary. Once an explosion occurs, it creates new excitations, amplifying local density and potentially triggering further explosions via the cascade mechanism of Paper I; spatial heterogeneity from recurrent cascades produces the concentration required for clump formation (theorem 4.2(c,d)). □

Remark 6.4 (Resolution of an open problem)**.** Equation (12) resolves, for the frequency factor, the open problem posed in an earlier draft of this paper (derive an explicit formula for the critical asymmetry). The remaining implicit dependence is the sustained co-density $\bar{\rho}$, which is determined by the channel asymmetry through the differential-depletion dynamics; deriving $\bar{\rho}(\mathcal{A}_c)$ in closed form remains open (section 8).

6.3 Hierarchy Depth Bound

Definition 6.5 (Hierarchy depth)**.** The *hierarchy depth* $H(\Sigma)$ of a configuration Σ is the maximum structural level realised:

$$H = \begin{cases} 0 & \text{if no excitations present,} \\ 1 & \text{if excitations present but no co-located pairs,} \\ 2 & \text{if co-located pairs exist but no bonds form,} \\ 3 & \text{if bonds form but are spatially isolated,} \\ 4 & \text{if spatially adjacent bonds form (clumps).} \end{cases}$$

Define $H^* = \max_n H(\Sigma(n))$, the peak hierarchy depth over the system's lifetime.

Proposition 6.6 (Hierarchy depth bound)**.** *The peak hierarchy depth satisfies:*

$$H^* \leq h(\mathcal{A}), \tag{13}$$

where $h : [0, 1] \to \{0, 1, 2, 3, 4\}$ is a non-decreasing step function with:

$h(\mathcal{A}) = 1$ *for $\mathcal{A} = 0$ (symmetric),*

$h(\mathcal{A}) \leq 3$ *for $0 < \mathcal{A} < \mathcal{A}_{\text{crit}}$ (subcritical; level 3 only by chance adjacency, typical depth 2),*

$h(\mathcal{A}) \leq 4$ *for $\mathcal{A} \geq \mathcal{A}_{\text{crit}}$ (supercritical).*

Proof. At $\mathcal{A} = 0$ (theorem 3.2), the hierarchy collapses to single-species annihilation: $H^* = 1$.

For $0 < \mathcal{A} < \mathcal{A}_{\text{crit}}$ (theorem 6.3(a)), bonds can form but no explosions occur, so no spatial concentration develops. Bonds are isolated: $H^* \leq 2$ (or 3 if adjacent bonds occur by chance, but without cascade-driven concentration this has vanishing probability for large P).

For $\mathcal{A} \geq \mathcal{A}_{\text{crit}}$ (theorem 6.3(b)), cascades produce spatial concentration, enabling the full hierarchy: $H^* = 4$ is achievable. □

Remark 6.7. The step function $h(\mathcal{A})$ captures the discrete, threshold-driven nature of the hierarchy. There are no "fractional" hierarchy levels: the system either achieves a level or does not. The transition is sharp at the level of the deterministic envelope (theorem 6.3); Poisson fluctuations smooth it into a narrow crossover window around $\mathcal{A}_{\text{crit}}$ in finite systems.

7 Complete Phase–Asymmetry Diagram

Combining the phase-structure results of Paper I with the present paper, the full behaviour of the model is classified by *two* control parameters: the initial energy density ε_0 and the asymmetry index $\mathcal{A}$.

ε_0	$\mathcal{A}$	Regime	Hierarchy
Low	Any	Immediate absorption	$H^* = 1$
High	$= 0$	Symmetric decay	$H^* = 1$
High	$< \mathcal{A}_{\text{crit}}$	Dissipative	$H^* \leq 3$ (typically 2)
High	$\geq \mathcal{A}_{\text{crit}}$	Cascade	$H^* \leq 4$

The *only* regime that produces the full structural hierarchy is the upper-right corner: high energy *and* sufficient asymmetry. Neither condition alone is sufficient.

8 Implications

8.1 Asymmetry as Constitutive, Not Contingent

The results of sections 3 to 5 establish that the two asymmetries in Paper I (the forbidden Y–Y channel and the frequency mismatch) are not modelling conveniences. They are *constitutive requirements* for the structural hierarchy. Removing either one collapses the hierarchy entirely.

This has a conceptual consequence: the minimum viable model for discrete cascade dynamics with emergent structure requires *at least two non-interchangeable species*. A single species, or two interchangeable species, cannot produce the four-level hierarchy.

8.2 Asymmetry Creates Timescale Separation

The frequency mismatch $\omega_X \neq \omega_Y$ creates two timescales in the interaction dynamics:

- A *fast* timescale $\tau_{\text{fast}} \sim 1/\omega_X^2$ governing X–X self-interaction (X depletes quickly),

- A *slow* timescale $\tau_{\text{slow}} \sim 1/(\omega_X \omega_Y)$ governing X–Y cross-interaction (Y persists longer).

This timescale separation is analogous to the fast-slow decomposition in singularly perturbed systems [Kuehn, 2015]. The "fast" X dynamics create the fluctuations; the "slow" Y dynamics provide the substrate. Without separation, both species fluctuate identically, and no structured behaviour emerges.

The interference factor adds a third, emergent timescale: the beat period $2\pi/|\omega_X - \omega_Y|$, which for small mismatch is much longer than either interaction timescale. The hierarchy of timescales (fast self-interaction, slow cross-interaction, slower beat) is what allows structure to accumulate between successive ripple re-excitations.

8.3 Differential Depletion as Symmetry Breaking

The forbidden Y–Y channel is a form of *explicit symmetry breaking*: it removes one interaction channel from species Y, making X and Y non-interchangeable. The differential depletion that follows (theorem 4.2) is a *dynamical consequence* of this explicit breaking.

The spatial concentration of bond sites (theorem 4.2(c,d)) is then a form of *spontaneous symmetry breaking*: the initial spatial distribution of X may be uniform, but the dynamics create non-uniform X density through stochastic fluctuation amplified by differential depletion. Bond sites "condense" at locations selected by noise, a discrete analogue of nucleation.

8.4 Connection to the Cascade Thesis

Paper I proved that the discrete cascade model reaches an absorbing state with probability one. This paper shows that the *richness* of the transient dynamics (the structural hierarchy, the complexity peak, the clump formation) depends critically on asymmetry. The three main theorems of Paper I (energy monotonicity, finite activity, absorption) hold regardless of $\mathcal{A}$. But the *interesting* behaviour (the part that makes the model more than a simple annihilation process) exists only for $\mathcal{A} > 0$. Figure 3 shows a hot, dense run (Paper I model) with both asymmetries: total energy decreases, X and Y evolve asymmetrically, and structure S and ripple F exhibit the transient complexity that vanishes in the symmetric limit.

8.5 Open Problems

1. **Optimal asymmetry.** For fixed initial energy $\mathcal{E}_{\text{tot}}(0)$, what value of $\mathcal{A}$ maximises the peak structural complexity $\max_n \mathcal{K}(n)$? Is there a unique optimum?

2. **Three or more species.** Can a three-species model (X, Y, Z) with richer asymmetry structure produce deeper hierarchies ($H^* > 4$)?

3. **Dynamic asymmetry.** What if ω_X and ω_Y are state-dependent (adaptive frequencies)? Can the system self-tune its asymmetry?

4. **Sustained co-density.** Theorem 6.3 gives the critical frequency asymmetry (12) in closed form, but the characteristic sustained co-density $\bar{\rho}$ appearing in it is determined implicitly by the differential-depletion dynamics. Derive $\bar{\rho}(\mathcal{A}_c, \rho_0, \varepsilon_0)$ explicitly, completing the closed-form phase boundary.

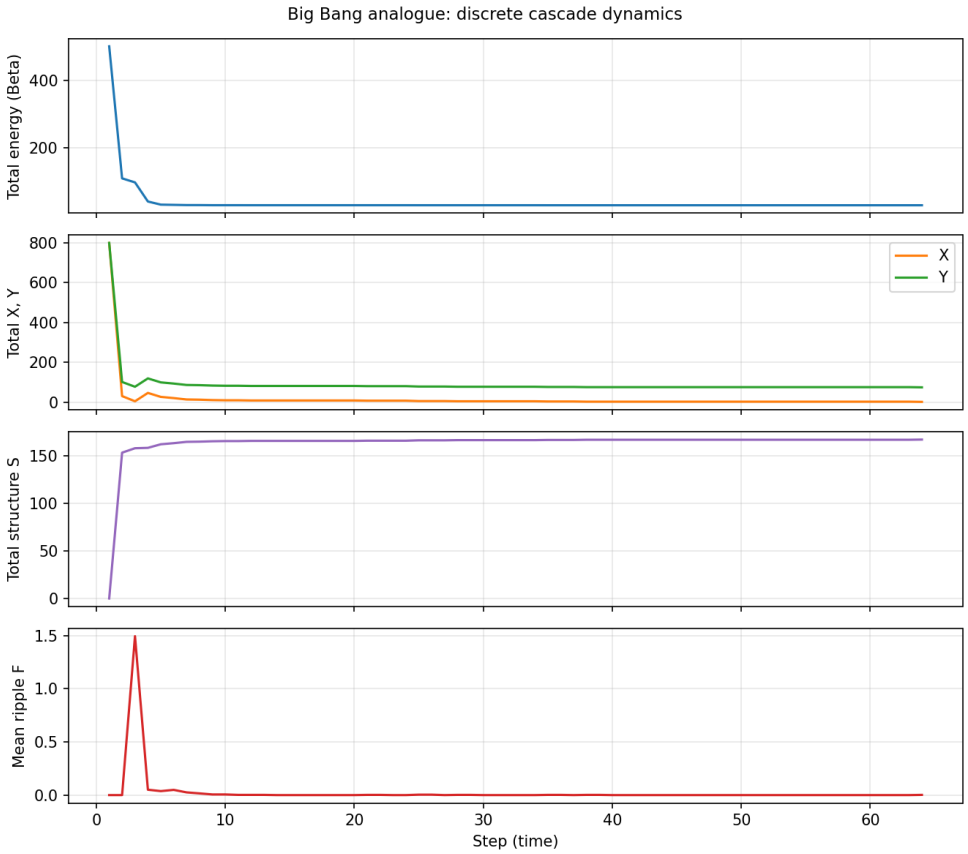

Figure 3: Paper I run with hot dense initial conditions ($P = 100$ sites): total energy (top), total X and Y (middle), total structure S and mean ripple F (bottom). Illustrates the transient dynamics that require $\mathcal{A} > 0$.

5. **Process-level formalisation.** The beat results are proved for the deterministic envelope of the increment process (theorem 5.1); upgrade them to statements about the full stochastic process (e.g., recurrence of threshold crossings with high probability under explicit conditions on the Poisson noise floor).

9 Conclusion

We have resolved the asymmetry question posed in the companion paper on discrete stochastic cascades. The answer is definitive:

1. The **forbidden Y–Y self-interaction channel** is necessary. It creates a differential depletion rate between species that drives minority-species condensation at bond sites. Without it, the depletion gap vanishes, the species ratio remains balanced, and the spatial concentration that underlies bond formation dissolves.

2. The **frequency mismatch** $\omega_X \neq \omega_Y$ is necessary. Through the phase-interference factor it generates a literal beat in the cross-channel efficiency, which is the sole recurrent deterministic source of ripple. Without it, the ripple decays with the depletion envelope, the system cannot sustain cascades, and no cascade-driven concentration develops.

3. The two asymmetries are **independently necessary**, and jointly (with sufficient initial energy and co-density) they **drive the full hierarchy**: when the asymmetry index $\mathcal{A}$ (the product of channel and frequency asymmetry) exceeds the critical threshold $\mathcal{A}_{\text{crit}}$ of theorem 6.3, the deterministic beat drive recurrently triggers cascades, and the full four-level hierarchy can emerge.

The structural hierarchy of the discrete cascade model (excitation $\rightarrow$ pair $\rightarrow$ bond $\rightarrow$ clump) is not an incidental feature. Within this model class, it is a direct, necessary consequence of species asymmetry, and it vanishes the moment that asymmetry is removed.

Broader context. The necessity of asymmetry for structural hierarchy connects to classical results on symmetry breaking in phase transitions [Goldenfeld, 1992] and to the theory of interacting particle systems and absorbing-state transitions [Liggett, 1985, Marro & Dickman, 1999, Krapivsky et al., 2010], but operates in a fundamentally different regime. In equilibrium statistical mechanics, symmetry breaking is *spontaneous*: a symmetric Hamiltonian gives rise to an asymmetric ground state. Here, the asymmetry is *constitutive*: it is built into the interaction rules as a design requirement (a forbidden self-interaction channel and a frequency mismatch), not an emergent property. Moreover, the system is finite, transient, and non-equilibrium; the standard machinery of infinite-volume limits and ergodic measures does not apply. The asymmetry index $\mathcal{A}$ and its critical threshold $\mathcal{A}_{\text{crit}}$ provide a quantitative bridge between this constitutive asymmetry and the structural complexity it enables, showing that the relationship is not merely qualitative but admits a threshold that is sharp at the level of the deterministic envelope.

Outlook. The principle that constitutive asymmetry is necessary for structural hierarchy may extend beyond the specific model class studied here. Any system in which multiple agent types interact through a shared, non-renewable resource faces the same fundamental question: does structural complexity require differentiation between types, or can identical agents produce it? The theorems in this paper answer that question definitively for one model class. Whether the answer generalises to broader classes of discrete stochastic systems, and whether the asymmetry index $\mathcal{A}$ has analogues in other settings, are questions we leave open for future work.

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